Generic Kármán-Rostovstev Plate Equations in an Affine Space

E. J. Brunelle*
U. S. Air Force Institute of Technology, Dayton, Ohio

The Kármán-Rostovstev plate equations are affinely transformed into equations that depend only on the generic plate constants D^*, H^*, and ϵ associated with the well-known linearized orthotropic plate/slab equations. Furthermore since H^* is a function of D^* and ϵ *only*, with ϵ being a weak parameter, *the only fundamental constant is D^**. This means that *regardless of what analysis technique is used, the results can be cast in terms of only one strong material parameter* (whose range is 0-1). The affine stretching is performed on the dependent variables (u,v,w,F) as well as the independent variables (x,y,t) so that the dependence of *all* quantities of interest on D^* and ϵ are easily calculated (e.g., even auxiliary quantities such as stress, strain, all weighted stress resultants, etc.). Some typical results (static side load response, collapse load limited by rib buckling) are presented by modifying an *approximate* technique due to Donnell.

Nomenclature

A,B,C	= affine stretching constants for the independent variables, Eqs. (18), (21a), and (21b)
a,b	= in-plane plate dimensions
a_0,b_0	= in-plane affine plate dimensions
$\mathfrak{D}$	= isotropic plate stiffness
D_{ij}	= orthotropic plate constants
D^*	= generalized rigidity, Eq. (28c)
E_{11},E_{22}	= in-plane elastic moduli
f	= affine stretching constant for the stress function, Eq. (14)
F,F_0	= stress function, affine stress function, Eq. (22g)
G_{12}	= in-plane shear modulus
H^*	= generalized extensional rigidity, Eq. (28b)
h	= plate thickness
K_1,K_2	= solution constants
$\mathcal{L}(\ ,\ )$	= two-parameter plate operator, Eq. (28c)
m	= mass per unit middle surface
p_0	= uniform affine side load
q	= physical side load
q_{11}	= first Fourier expansion coefficient of q
S,S_I	= global solution scalars
S_{x_0},S_{y_0}	= affine (average) membrane stresses
S_u	= ultimate compressive load parameter, Eq. (53)
t,t_0	= time, affine time
$u,v,w;u_0,v_0,w_0$	= displacements, affine displacements
w_{11}	= first Fourier expansion coefficient of w_0
$x,y;x_0,y_0$	= space dimensions, affine space dimensions
α,β,γ	= affine stretching constants for the displacements, Eqs. (20a), (20b), (21a), (21b), and (28a)
ϵ	= generalized Poisson's ratio, Eq. (28d)
ν_{12}	= Poisson's ratio
$(\sigma_{\rm rib})_0$	= affine rib buckling stress
Σ_0	= rib buckling load parameter, Eq. (54)

Received May 1, 1984; presented as Paper 84-0891 at the AIAA/ASME/ASCE/AHS 25th Structures, Structural Dynamics and Materials Conference, Palm Springs, CA, May 14-16, 1984; revision received March 13, 1985.

*Professor, Department of Aeronautics and Astronautics; currently at the Rensselaer Polytechnic Institute, Troy, NY. Member AIAA.

Introduction

THE elegant and imposing equations of von Kármán,[1] as modified by Rostovstev,[2] can be solved approximately by a variety of techniques, all of which yield specialized results (i.e., for the given set of elastic constants, specific relational curves such as deflection vs side load for a fixed aspect ratio, etc., can be solved) in spite of being generally laborious. A standard summary of such techniques is given by Chia.[3] So little solution information exists that it is not possible to give a general overview of how the solutions change when specific sets (usually the *unit* set) of elastic constants change.† That is, a *physical intuition* concerning the solutions has yet to be abstracted from specific results. Clearly, the existence of eight elastic constants in the orthotropic‡ Kármán-Rostovstev (K-R) equations makes it very difficult to predict the parametric solution behavior and, in fact, the general question "if *these* constants are changed and *those* constants are not changed, how is the specific solution (static, dynamic, postbuckling, etc.) altered?" does not presently have a satisfactory answer.

A method that allows solutions in terms of significantly fewer plate constants should be very valuable. [Indeed, such has been the case for the linearized orthotropic (uncoupled) plate/slab equations as presented by Brunelle[4-8] (and with Oyibo[9,10]); this material provides an overview of the language and concepts of such linearized problems and is recommended adjunct reading.]

It is natural to ask if the use of generic constants and affine transformations will recast the K-R equations into an analytically more attractive set of relations. The answer, as will be demonstrated, is *yes* and the generic constants involved are the *same* constants (D^*, H^*, and ϵ) that appear in the linearized orthotropic plate/slab equations. However, the required affine space transformations are considerably more complicated, since the dependent variables u, v, w, and F (displacements and stress function) must be transformed as well as the independent variables x, y, and t. Furthermore, since H^* is a function of D^* and ϵ and since solutions are weakly dependent on ϵ, there is only one strong elastic variable (D^*) in the *entire transformed equation* set (the K-R equations, in-plane stress-strain relations, and w-related boundary conditions). This fact underlines the basic contribution of the present work. It is now possible to acquire a comprehensive grasp of the K-R equation *solution*

†A more orderly state of affairs exists with the isotropic Kármán equations since E and ν are only independent elastic constants [i.e., $G = E/2(1+\nu)$].

‡The anisotropic version of the Kármán-Rostovstev equation is much more complicated and is left for future investigations.

dependence on the elastic constants because 1) whatever analysis technique§ is used (series, perturbation, MWR, etc.) the answers appear in terms of only D^*, ϵ, and the affine aspect ratio a_0/b_0, and 2) since D^* has the limited range of 0-1 *for all materials*,¶ it is truly feasible to display the entire range of parametric solution effects.

The K-R Equations and Their Transformed Counterparts

Using the affine transformations

$$x = Ax_0 \tag{1a}$$

$$y = By_0 \tag{1b}$$

$$t = Ct_0 \tag{1c}$$

$$u = \alpha u_0 \tag{1d}$$

$$v = \beta v_0 \tag{1e}$$

$$w = \gamma w_0 \tag{1f}$$

$$F = fF_0 \tag{1g}$$

the K-R equations

$$\frac{1}{E_{22}}\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_{11}}\frac{\partial^4 F}{\partial y^4} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{2}$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + m\frac{\partial^2 w}{\partial t^2} = q(x,y,t) + h\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) \tag{3}$$

and the stress-strain relations

$$\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{1}{E_{11}}\left(\frac{\partial^2 F}{\partial y^2} - \nu_{12}\frac{\partial^2 F}{\partial x^2}\right) \tag{4}$$

$$\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{E_{22}}\left(-\nu_{12}\frac{E_{22}}{E_{11}}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2}\right) \tag{5}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = -\left(\frac{1}{G_{12}}\right)\frac{\partial^2 F}{\partial x \partial y} \tag{6}$$

become, after multiplying Eq. (2) by $A^2B^2\sqrt{E_{11}E_{22}}$ and dividing Eq. (3) by γ,

$$\frac{B^2}{A^2}\sqrt{\frac{E_{11}}{E_{22}}}\frac{\partial^4 F_0}{\partial x_0^4} + 2H^*\frac{\partial^4 F_0}{\partial x_0^2 \partial y_0^2} + \frac{A^2}{B^2}\sqrt{\frac{E_{22}}{E_{11}}}\frac{\partial^4 F_0}{\partial y_0^4} = \frac{\gamma^2}{f}\sqrt{E_{11}E_{22}}\left[\left(\frac{\partial^2 w_0}{\partial x_0 \partial y_0}\right)^2 - \frac{\partial^2 w_0}{\partial x_0^2}\frac{\partial^2 w_0}{\partial y_0^2}\right] \tag{7}$$

$$\frac{D_{11}}{A^4}\frac{\partial^4 w_0}{\partial x_0^4} + 2D^*\frac{\sqrt{D_{11}D_{22}}}{A^2B^2}\frac{\partial^4 w_0}{\partial x_0^2 \partial y_0^2} + \frac{D_{22}}{B^4}\frac{\partial^4 w_0}{\partial y_0^4} + \frac{m}{C^2}\frac{\partial^2 w_0}{\partial t_0^2} = \frac{q}{\gamma} + \frac{hf}{A^2B^2}\left(\frac{\partial^2 F_0}{\partial y_0^2}\frac{\partial^2 w_0}{\partial x_0^2} - 2\frac{\partial^2 F_0}{\partial x_0 \partial y_0}\frac{\partial^2 w_0}{\partial x_0 \partial y_0} + \frac{\partial^2 F_0}{\partial x_0^2}\frac{\partial^2 w_0}{\partial y_0^2}\right) \tag{8}$$

$$\alpha A\frac{\partial u_0}{\partial x_0} + \frac{1}{2}\gamma^2\left(\frac{\partial w_0}{\partial x_0}\right)^2 = \frac{f}{E_{11}}\left(\frac{A^2}{B^2}\frac{\partial^2 F_0}{\partial y_0^2} - \nu_{12}\frac{\partial^2 w_0}{\partial x_0^2}\right) \tag{9}$$

$$\beta B\frac{\partial v_0}{\partial y_0} + \frac{1}{2}\gamma^2\left(\frac{\partial w_0}{\partial y_0}\right)^2 = \frac{f}{E_{22}}\left(-\nu_{12}\frac{E_{22}}{E_{11}}\frac{\partial^2 F_0}{\partial y_0^2} + \frac{B^2}{A^2}\frac{\partial^2 F_0}{\partial x_0^2}\right) \tag{10}$$

$$\beta B\frac{\partial v_0}{\partial x_0} + \alpha A\frac{\partial u_0}{\partial y_0} + \gamma^2\frac{\partial w_0}{\partial x_0}\frac{\partial w_0}{\partial y_0} = -\frac{f}{G_{12}}\frac{\partial^2 F_0}{\partial x_0 \partial y_0} \tag{11}$$

(In addition Eqs. (9-11) have been multiplied by A^2, B^2, and AB, respectively.)

Demanding that $D_{11}/A^4 = 1$ and $D_{22}/B^4 = 1$, it is seen that

$$A^4 = D_{11} \tag{12a}$$

$$B^4 = D_{22} \tag{12b}$$

$$A^2B^2 = \sqrt{D_{11}D_{12}} \tag{12c}$$

$$\frac{B^2}{A^2} = \sqrt{\frac{D_{22}}{D_{11}}}\left(\text{and } \frac{A^2}{B^2} = \sqrt{\frac{D_{11}}{D_{22}}}\right) \tag{12d}$$

$$\sqrt{\frac{E_{11}}{E_{22}}}\frac{B^2}{A^2} = \sqrt{\frac{E_{22}}{E_{11}}}\frac{A^2}{B^2} \equiv 1 \tag{12e}$$

Now concentrating for the moment on K-R equations (7-8), it is seen that a simple form of these equations will appear if

$$(\gamma^2/f)\sqrt{E_{11}E_{22}} = 1 \tag{13a}$$

and

$$hf/A^2B^2 = 1 \tag{13b}$$

Using Eq. (12c) in Eq. (13b) yields

$$f = \sqrt{D_{11}D_{22}}/h \tag{14}$$

and then from Eq. (13a)

$$\gamma^2 = h^2/12[1-(\epsilon D^*)^2] \tag{15}$$

At this point, the K-R equations are in the desired form (they will be displayed shortly with the complete affinely transformed set), the affine stretching constants for x and y (that is, A and B) have been determined by Eqs. (12a) and (12b), and the affine stretching for w and F [Eqs. (1f) and 1g)] have been determined by Eqs. (14) and (15). That is,**

$$w = [h/\sqrt{12[1-(\epsilon D^*)^2]}]\, w_0 \tag{16}$$

$$F = [\sqrt{D_{11}D_{22}}/h]F_0 \tag{17}$$

§That is, solution methods must be provided; the results of this paper *must not be misinterpreted* as results to be used with existing solutions.

¶This has been stated as a law by the author,[7] i.e., there is no contradiction for all known specially orthotropic materials. The behavior of the constant *related* to D^* as a function of angle with respect to the principle axis is discussed in the Appendix.

**It is noted that for other fixed quantities, $F \sim wh^2$ is seen from the ratio of Eqs. (16) and (17).

It remains to determine the in-plane displacement (and time) stretching constants. The time-stretching constant is easily seen to be

$$C^2 = m \tag{18}$$

so that the affine stretching relation for the time variable is given by

$$t = \sqrt{m}t_0 \tag{19}$$

Thus, attention turns to Eqs. (9-11) to find the values for α and β. A close inspection of the structure of these equations suggests trying substitutions of the form,

$$\alpha = \gamma^2/K_1 \tag{20a}$$

and

$$\beta = \gamma^2/K_2 \tag{20b}$$

Some algebra will reveal that

$$K_1 = D_{11}^{1/4} = A \tag{21a}$$

$$K_2 = D_{22}^{1/4} = B \tag{21b}$$

and that the associated stress-strain equations take on a particularly simple form. Thus the complete set of affinely transformed equations corresponding to Eqs. (1-3) is given by

$$x = D_{11}^{1/4} x_0 \tag{22a}$$

$$y = D_{22}^{1/4} y_0 \tag{22b}$$

$$t = m^{1/2} t_0 \tag{22c}$$

$$u = [\gamma^2/(D_{11})^{1/4}]u_0 \tag{22d}$$

$$v = [\gamma^2/(D_{22})^{1/4}]v_0 \tag{22e}$$

$$w = \gamma w_0 \tag{22f}$$

$$F = (\sqrt{D_{11}D_{22}}/h)F_0 \tag{22g}$$

$$\frac{\partial^4 F_0}{\partial w_0^4} + 2H^* \frac{\partial^4 F_0}{\partial w_0^2 \partial y_0^2} + \frac{\partial^4 F_0}{\partial y_0^4} + \frac{1}{2}\mathcal{L}(w_0, w_0) = 0 \tag{23}$$

$$\frac{\partial^4 w_0}{\partial x_0^4} + 2D^* \frac{\partial^4 w_0}{\partial x_0^2 \partial y_0^2} + \frac{\partial^4 w_0}{\partial y_0^4} + \frac{\partial^2 w_0}{\partial t_0^2}$$

$$= (1/\gamma) q(x_0, y_0, t_0) + \mathcal{L}(F_0, w_0) \tag{24}$$

$$\frac{\partial u_0}{\partial x_0} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x_0}\right)^2 = \frac{\partial^2 F_0}{\partial y_0^2} - \epsilon D^* \frac{\partial^2 F_0}{\partial x_0^2} \tag{25}$$

$$\frac{\partial v_0}{\partial y_0} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y_0}\right)^2 = -\epsilon D^* \frac{\partial^2 F_0}{\partial y_0^2} + \frac{\partial^2 F_0}{\partial x_0^2} \tag{26}$$

$$\frac{\partial v_0}{\partial x_0} + \frac{\partial u_0}{\partial y_0} + \frac{\partial w_0}{\partial x_0}\frac{\partial w_0}{\partial y_0} = -2(H^* + \epsilon D^*)\frac{\partial^2 F_0}{\partial x_0 \partial y_0} \tag{27}$$

where

$$\gamma^2 = h^2/\{12[1 - (\epsilon D^*)^2]\} \tag{28a}$$

$$H^* = \left(\frac{1}{2G_{12}} - \frac{\nu_{12}}{E_{11}}\right)\sqrt{E_{11}E_{22}} \tag{28b}$$

$$D^* = (D_{12} + 2D_{66})/\sqrt{D_{11}D_{22}} \tag{28c}$$

$$\epsilon D^* = D_{12}/\sqrt{D_{11}D_{22}} \tag{28d}$$

$$\mathcal{L}(\ ,\ ) = \frac{\partial^2}{\partial y_0^2}\frac{\partial^2}{\partial x_0^2} - 2\frac{\partial^2}{\partial x_0 \partial y_0}\frac{\partial^2}{\partial x_0 \partial y_0} + \frac{\partial}{\partial x_0^2}\frac{\partial^2}{\partial y_0^2} \tag{28e}$$

and H^* is related to D^* and ϵ through the relation

$$1/H^* = D^*(1-\epsilon)/[1 - \epsilon(D^*)^2] \tag{29}$$

Equation (29) is displayed in Fig. 1. Since the solutions are weakly dependent on ϵ, only one strong elastic parameter (say D^*) exists for these equations. (This weak dependence is based on accumulated experience with various solution categories.)

Thus, global solution scalars (e.g., frequencies, collapse loads, extremum deflections), say S, are usually expressed as††

$$S = S(a_0/b_0, D^*, \epsilon, p) \tag{30a}$$

Recalling that solution scalars (S_I) for the isotropic Kármán equations are of the form

$$S_I = S_I(a/b, \mathfrak{D}, \nu, p) \tag{30b}$$

it must be concluded that, if a physical intuition can be abstracted for the isotropic Kármán plate problem, then it most certainly also can be abstracted for the affinely transformed K-R equations. In the next section, some very simple examples are used to show how clearly the parametric dependence of solution scalars may be displayed.

Approximate General Solutions for SS-SS-SS-SS Plates

Using the *simplest possible* truncated series solution (the isotropic equivalent of this technique is summarized‡‡ in Donnell[11]), it is assumed that

$$q(x_0, y_0) = q_{11}(\sin \pi x_0/a_0)(\sin \pi y_0/b_0) \tag{31a}$$

$$u_0 = u_{00} + u_1 x_0 + u_{20}\sin 2\pi x_0/a_0 \tag{31b}$$

$$v_0 = v_{00} + v_1 y_0 + v_{02}\sin 2\pi y_0/b_0 \tag{31c}$$

$$w_0 = w_{11}(\sin \pi x_0/a_0)(\sin \pi y_0/b_0) \tag{31d}$$

$$F_0 = S_{x_0} y_0^2/2 + S_{y_0} x_0^2/2 + f_{20}\cos 2\pi x_0/a_0 + f_{02}\cos 2\pi y_0/b_0 \tag{31e}$$

and the identities

$$2\cos^2 \pi z_0/c_0 = 1 + \cos 2\pi z_0/c_0 \tag{32a}$$

and

$$2\sin^2 \pi z_0/c_0 = 1 - \cos 2\pi z_0/c_0 \tag{32b}$$

††p is a miscellaneous parameter such as side load parameter, rib buckling stress, yield stress, etc. Also, depending on the boundary conditions, ϵ (and hence ν in the isotropic problem) may *not* be present in the solution.

‡‡Note that Donnell's primary intent was not for solution accuracy; in general, the over-riding interest was solution details. The present work carries on in that tradition. In this same vein, note that the assumed form of F_0 makes $\partial^4 F_0/\partial x_0^2 \partial y_0^2$ identically zero; therefore, the effect of H^* in Eq. (23) is lost in this simplest of approximations. Such would not be the case with more sophisticated solution techniques.

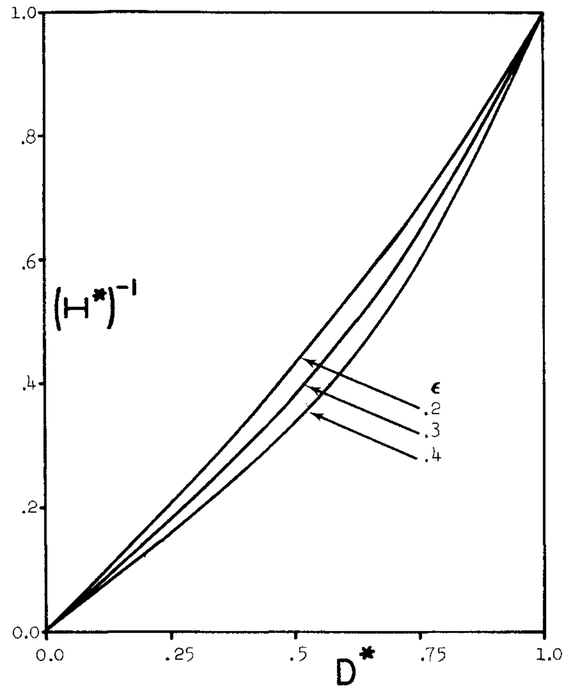

Fig. 1 H^* inverse vs D^* with ϵ as a parameter.

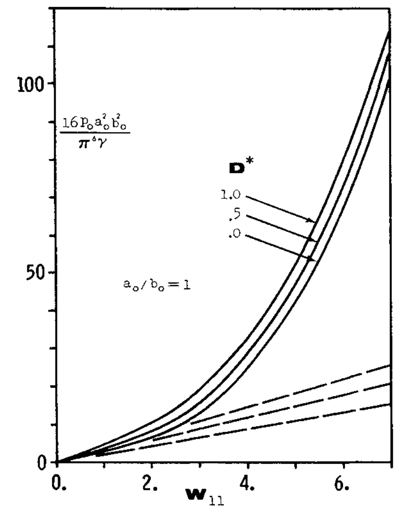

Fig. 2 Load parameter $16p_0a_0^2b_0^2/\pi^6\gamma$ vs the deflection parameter w_{11} for three values of D^* ($a_0/b_0=1$ and the sides free to pull inward).

will be used freely. (z_0/c_0 can be either x_0/a_0 or y_0/b_0.) From Eq. (23), it is concluded that

$$f_{20}=\frac{w_{11}^2}{32}\left(\frac{a_0}{b_0}\right)^2 \tag{33a}$$

$$f_{02}=\frac{w_{11}^2}{32}\left(\frac{b_0}{a_0}\right)^2 \tag{33b}$$

and these results inserted into Eq. (24) will demand that

$$w_{11}\frac{\pi^4}{a_0^2b_0^2}\left[\left(\frac{b_0}{a_0}\right)^2+2D^*+\left(\frac{a_0}{b_0}\right)^2\right]+w_{11}S_{x_0}\left(\frac{\pi}{a_0}\right)^2$$
$$+w_{11}S_{y_0}\left(\frac{\pi}{b_0}\right)^2+\frac{w_{11}^3}{8}\left[\left(\frac{\pi}{a_0}\right)^4+\left(\frac{\pi}{b_0}\right)^4\right]=q_{11} \tag{34}$$

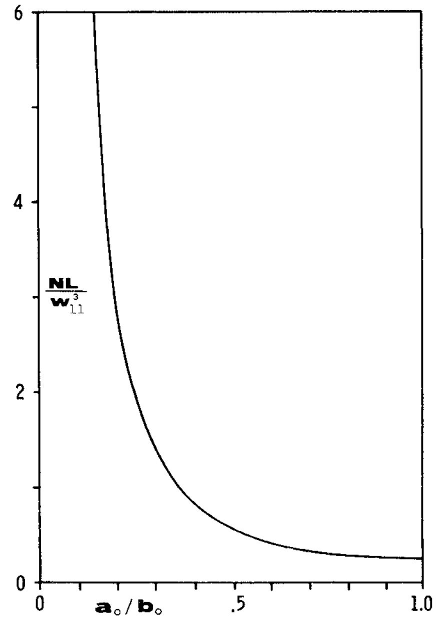

Fig. 3 Nonlinear coefficient NL/w_{11}^3 vs a_0/b_0 (the sides free to pull inward).

Additionally, Eqs. (25-27) yield the following results§§ about the membrane displacements:

$$u_1+\frac{w_{11}^2}{8}\left(\frac{\pi}{a_0}\right)^2=S_{x_0}-\epsilon D^*S_{y_0} \tag{35}$$

$$v_1+\frac{w_{11}^2}{8}\left(\frac{\pi}{b_0}\right)^2=S_{y_0}-\epsilon D^*S_{x_0} \tag{36}$$

and

$$(u_{20})\frac{2\pi}{a_0}+\frac{w_{11}^2}{8}\left(\frac{\pi}{a_0}\right)^2=\epsilon D^*f_{20}\left(\frac{2\pi}{a_0}\right)^2 \tag{37}$$

$$(v_{02})\frac{2\pi}{b_0}+\frac{w_{11}^2}{8}\left(\frac{\pi}{b_0}\right)^2=\epsilon D^*f_{02}\left(\frac{2\pi}{b_0}\right)^2 \tag{38}$$

The solutions basically reside in Eq. (34) with Eqs. (35-38) being used for auxiliary results, especially for the satisfaction of membrane boundary conditions. In the following section, specific problems are addressed.

Plates with Uniform Lateral Affine Load p_0 (Edges Free to Pull Inward)

Considering q_{11} to be the leading coefficient in a double Fourier expansion for a uniform affine load p_0, it is easily verified that

$$q_{11}=16p_0/\pi^2 \tag{39}$$

Therefore, with the edges free to pull inward,¶¶ $S_{x_0}=S_{y_0}=0$ and Eq. (34) with Eq. (39) inserted becomes

$$\frac{16p_0a_0^4}{\gamma\pi^6}=\left[1+2D^*\left(\frac{a_0}{b_0}\right)^2+\left(\frac{a_0}{b_0}\right)^4\right]w_{11}$$
$$+\frac{1}{8}\left[1+\left(\frac{a_0}{b_0}\right)^4\right]w_{11}^3 \tag{40}$$

§§Equation (27), to first order, is identically satisfied, hence the nonappearance of the $(H^*+\epsilon D^*)$ coefficient.

¶¶Free is to be interpreted as an *integrated* average (i.e., $S_{x_0}=S_{y_0}=0$).

or if a more symmetric form is desired Eq. (40) may be written as

$$\frac{16p_0a_0^2b_0^2}{\gamma\pi^6}=\left[\left(\frac{b_0}{a_0}\right)^2+2D^*+\left(\frac{a_0}{b_0}\right)^2\right]w_{11}$$
$$+\frac{1}{8}\left[\left(\frac{a_0}{b_0}\right)^2+\left(\frac{b_0}{a_0}\right)^2\right]w_{11}^3 \tag{41}$$

This last form is particularly interesting, since it is invariant with respect to the inversion of the quantity a_0/b_0. Obviously, then, it is only necessary to find answers in the region $0<a_0/b_0\leq 1$; thus, another solution economy has been utilized. In either form, it is noted that the *nonlinear part* of the solution depends *only** on w_{11} and a_0/b_0, thus letting NL denote the nonlinear portion of the solution,

$$\text{NL}=\frac{1}{8}\left[\left(\frac{a_0}{b_0}\right)^2+\left(\frac{b_0}{a_0}\right)^2\right]w_{11}^3 \tag{42}$$

and, furthermore, since the nonlinear effect of w_{11} is simply the *multiplier* w_{11}^3, the final level of simplicity would be to write

$$\frac{\text{NL}}{w_{11}^3}=\frac{1}{8}\left[\left(\frac{a_0}{b_0}\right)^2+\left(\frac{b_0}{a_0}\right)^2\right] \tag{43}$$

Finally, since the nonlinear coefficient is simply related to the linear coefficient, their ratio may be of interest. This ratio R is given by

$$R=1/\{8[1+2D^*/AB]\} \tag{44}$$

where AB is given by

$$AB=\left(\frac{a_0}{b_0}\right)^2+\left(\frac{b_0}{a_0}\right)^2 \tag{45}$$

Thus, this ratio is smallest when $a_0/b_0=1$, i.e., $R=1/[8(1+D^*)]$ and, in the limit of $a_0/b_0\to 0$ (or $a_0/b_0\to\infty$), the coefficient is given by $R=1/8$; that is, there exists *at most* a factor of two difference in these two extreme cases.

Figure 2 presents a traditional curve of the load parameter vs the maximum deflection [from Eq. (41)] for $a_0/b_0=1$ with the linear solution shown in dashed lines. Figure 3 presents the nonlinear coefficient NL/w_{11}^3 vs a_0/b_0 from Eq. (43).

Plates with Uniform Lateral Affine Load p_0 (Edges Restrained from Pulling Inward)

Imposing the in-plane boundary conditions that $u_0=0$ at $x_0=0$, a_0 and $v_0=0$ at $y_0=0$, b_0 yields $u_{00}=v_{00}=u_1=v_1=0$ and the following values of the average membrane affine stress:

$$S_{x_0}=\frac{1}{8}w_{11}^2\left(\frac{\pi}{a_0}\right)^2\left[\frac{1+\epsilon D^*(a_0/b_0)^2}{1-(\epsilon D^*)^2}\right] \tag{46}$$

$$S_{y_0}=\frac{1}{8}w_{11}^2\left(\frac{\pi}{b_0}\right)^2\left[\frac{1+\epsilon D^*(b_0/a_0)^2}{1-(\epsilon D^*)^2}\right] \tag{47}$$

*At this point, the statement cannot be generalized to higher-order solutions. The educated guess, of course, is that the higher-order terms to the solution will depend on D^*, but in a convergent series the higher-order terms provide weaker and weaker *overall* corrections to the first-order solution.

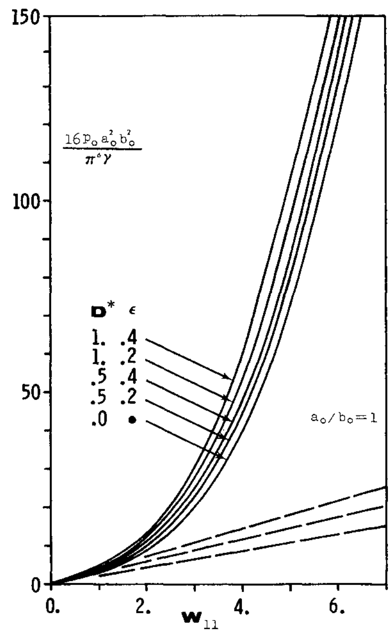

Fig. 4 Load parameter $16p_0a_0^2b_0^2/\pi^6\gamma$ vs the deflection parameter w_{11} for three values of D^* (and two ϵ values) ($a_0/b_0=1$ and the sides restrained from pulling inward).

and the in-plane affine displacements

$$u_0=u_{20}\sin 2\pi x_0/a_0 \tag{48a}$$

$$v_0=v_{02}\sin 2\pi y_0/b_0 \tag{48b}$$

where u_{20} and v_{02} are given by Eqs. (37) and (38) in conjunction with Eqs. (33).

Inserting Eqs. (39), (46), and (47) into Eq. (34) yields the desired result (once again invariant upon the inversion of a_0/b_0), namely

$$\frac{16p_0a_0^2a_0^2}{\gamma\pi^6}=\left[\left(\frac{a_0}{b_0}\right)^2+2D^*+\left(\frac{b_0}{a_0}\right)^2\right]w_{11}$$
$$+\left\{\left[\left(\frac{a_0}{b_0}\right)^2+2\epsilon D^*+\left(\frac{b_0}{a_0}\right)^2\right][1-(\epsilon D^*)^2]^{-1}\right.$$
$$\left.+\left(\frac{a_0}{b_0}\right)^2+\left(\frac{b_0}{a_0}\right)^2\right\}\left(\frac{w_{11}}{2}\right)^3 \tag{49}$$

Letting NL1 be the nonlinear portion of the solution, it is seen that

$$\text{NL1}/w_{11}^3=g(a_0/b_0,\epsilon D^*) \tag{50}$$

Figure 4 presents a traditional curve of the load parameter vs the maximum deflection [from Eq. (49)] for $a_0/b_0=1$ with the linear solution shown in dashed lines. Figure 5 presents the nonlinear coefficient $\text{NL1}/w_{11}^3$ vs a_0/b_0 for two values of ϵD^* from Eq. (50) and Fig. 6 presents a traditional curve of the load parameter vs a_0/b_0 for $w_{11}=3$, where again the linear solution is shown in dashed lines.

Ultimate Plate Buckling Load (Limited by Rib Buckling)

Assuming that the edges are free to move in the plane of the panel ($S_{y_0}=0$) and since there is no lateral load ($q_{11}=0$)

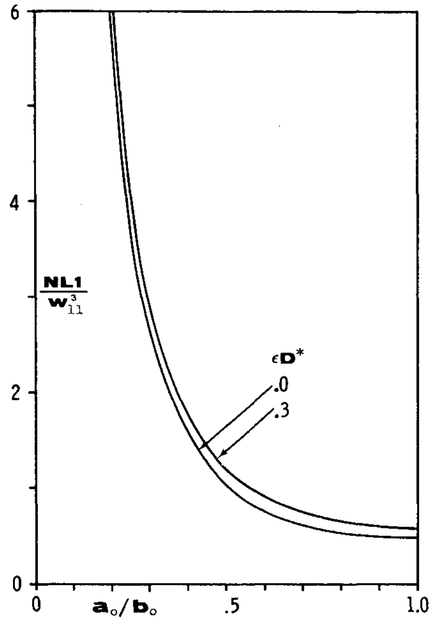

Fig. 5 Nonlinear coefficient $NL1/w_{11}^3$ vs a_0/b_0 for two values of ϵD^* (the sides restrained from pulling inward).

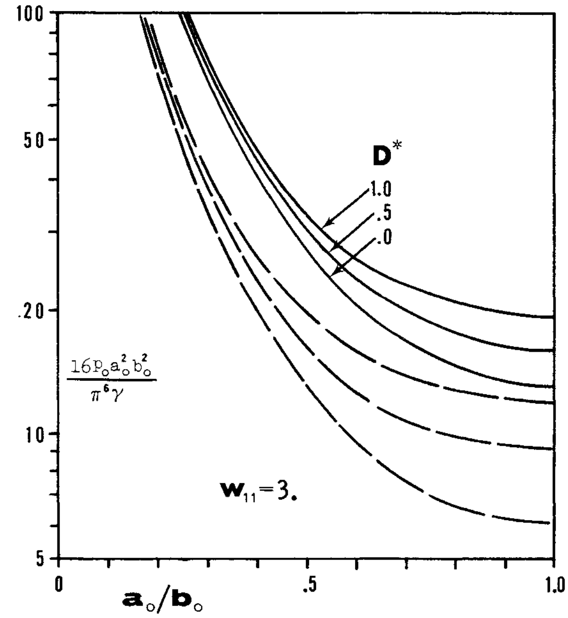

Fig. 6 Load parameter $16p_0a_0^2/b_0^2/\pi^6\gamma$ vs a_0/b_0 for $w_{11}=3$ (the sides restrained from pulling inward).

Eq. (34), upon using the result†

$$\left(\frac{\pi}{a_0}\right)^2\frac{w_{11}^2}{8}=S_x+(\sigma_r)_0 \tag{51}$$

becomes

$$\frac{-S_u}{\sqrt{\Sigma_0}}=\left\{\frac{1}{\sqrt{\Sigma_0}}\left[\left(\frac{a_0}{b_0}\right)^2+2D^*+\left(\frac{b_0}{a_0}\right)^2\right]+\left[\left(\frac{a_0}{b_0}\right)^2+\left(\frac{b_0}{a_0}\right)^2\right]\sqrt{\Sigma_0}\right\}\Big/\left\{\left(\frac{a_0}{b_0}\right)^2+2\left(\frac{b_0}{a_0}\right)^2\right\} \tag{52}$$

where the definitions

$$S_u=S_{x_0}(a_0/\pi)^2 \tag{53}$$

and

$$\Sigma_0=(\sigma_{\rm rib})_0(a_0/\pi)^2 \tag{54}$$

have been utilized. In particular, if $a_0/b_0=1$, Eq. (52) reduces to a much simpler form, namely

$$\frac{-S_u}{\sqrt{\Sigma_0}}=\frac{2}{3}\left[\frac{1}{\sqrt{\Sigma_0}}(1+D^*)+\sqrt{\Sigma_0}\right] \tag{55}$$

Figure 7 plots an ultimate load parameter curve similar to traditional curves presented for isotropic Kármán plates. In particular, note the isotropic Kármán solution added for comparison.

Generalization of the Results for Cross-Ply Symmetric Plates

Using results developed by Tsai,[12] rewriting Eq. (2) in terms of compliances, and using the ideas described by the author,[7,8] all of the above results can be modified slightly to describe cross-ply symmetric laminated K-R plates.

†$\partial^2F_0/\partial y_0^2$ evaluated at $y_0=0$, b_0 *is* $-(\sigma_r)_0$ for collapse limited by rib buckling.

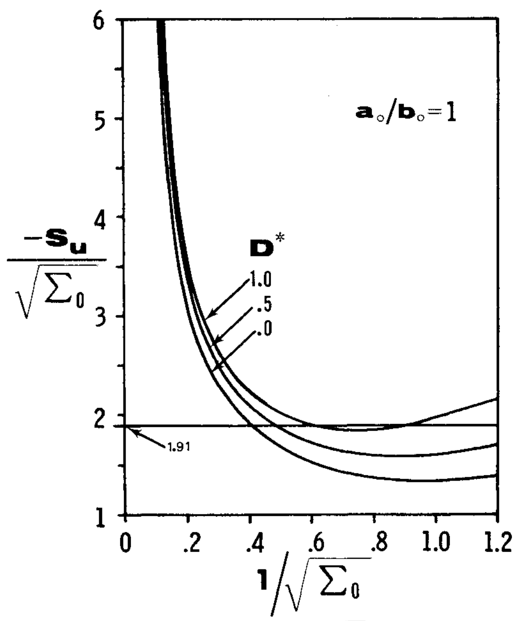

Fig. 7 Ultimate load parameter $-S_u/\sqrt{\Sigma_0}$ vs the inverse of the rib load parameter Σ_0 for three values of D^*, $a_0/b_0=1$ (collapse load is limited by rib buckling).

Conclusions

The K-R equations have been presented in a particularly useful affine space and their *solution properties* have been examined by a series of problems. The problems are very simple, but they serve to illustrate the usefulness of this paper's central contribution—namely, that a set of coupled, moderate rotation, orthotropic plate equations (which can be easily modified to include cross-ply symmetric composite plates) in just one strong elastic parameter (D^*) is available for analysis and design purposes.

Appendix: $\bar{D}^*$ for an Anisotropic Lamina

Among several quantities associated with D^* for off-axis directions (all related by angle-dependent scale factors), a convenient choice is $\bar{D}^*$ defined as

$$\bar{D}^* = (\bar{D}_{12} + 2\bar{D}_{66})/\sqrt{D_{11}D_{22}} \tag{A1}$$

In particular, note that the denominator contains the principal values of D_{11} and D_{22}, i.e., they are *not* barred quantities.

Given the lamina definitions for $\bar{D}_{12}$ and $\bar{D}_{66}$, namely,

$$\bar{D}_{12} = (D_{11} + D_{22} - 4D_{66})\sin^2\theta\cos^2\theta + D_{12}(\sin^4\theta + \cos^4\theta) \tag{A2}$$

and

$$\bar{D}_{66} = (D_{11} + D_{22} - 2D_{12} - 2D_{66})\sin^2\theta\cos^2\theta + D_{66}(\sin^4\theta + \cos^4\theta) \tag{A3}$$

some manipulation using trigonometric identities reveals a particularly simple form for $\bar{D}^*$. It is expressed as

$$\bar{D}^* = D^* + \tfrac{3}{4}[K^{-1} + K - 2D^*]\sin^2 2\theta \tag{A4}$$

where

$$D^* = (D_{12} + 2D_{66})/\sqrt{D_{11}D_{22}} \tag{A5}$$

and

$$K = \sqrt{D_{22}/D_{11}} \tag{A6}$$

The difference $\Delta\bar{D}^* \equiv \bar{D}^* - D^*$ depends only on a K- and D^*-dependent coefficient [namely, 0.75 $(K^{-1} + K - 2D^*)$] and a simple trigonometric function, $\sin^2 2\theta$. The angles for maximum $\Delta\bar{D}^*$ and $\bar{D}^*$ are $\pm\pi/4$ and $\pm 3\pi/4$. The dependence of $\Delta\bar{D}^*$ on D^* and K is obvious.

However, the most important feature of Eq. (A4) is that its overall statement, which is

$$\bar{D}^* \geq D^* \tag{A7}$$

does not in any way infringe on the statement

$$0 \leq D^* \leq 1 \tag{A8}$$

Indeed, the above statements are uncoupled from one another in the sense that, although $\bar{D}^*$ depends on D^*, it has no influence on the *range* of D^* previously established by the author.

Acknowledgments

This research was carried out at the Rensselaer Polytechnic Institute under a joint NASA/AFOSR Grant NGL 33-018-033; these agencies are thanked for their generous support and encouragement. It is also a pleasure to thank Ms. Amy Moore for her competent typing skills.

References

[1]von Kármán, T., "Festigkeitsprobleme im Mashinenbau," *Encykl. der Mathematischen Wissenschaften*, Vol. 4, No. 4, 1910, pp. 348-352.

[2]Rostovstev, G. G., "Calculation of Thin Plane Panels Reinforced by Stiff Ribs," *Proceedings of the Leningrad Institute for Engineers of the Civil Air Fleet*, No. 20, 1940 (in Russian).

[3]Chia, C. Y., *Nonlinear Analysis of Plates*, McGraw-Hill Book Co., New York, 1980, pp. 38-45.

[4]Brunelle, E. J., "The Use of a Double Affine Transformation to Obtain a Class of Generalized Elastodynamic Solutions for Orthotropic Plates," Paper presented at SIAM 1980 National Meeting, Alexandria, VA, June 1980.

[5]Brunelle, E. J., "The Use of Affine Transformations in the Solution of Composite Structures Problems," Paper presented at 17th Annual Meeting of Society of Engineering Science, Atlanta, Dec. 1980.

[6]Brunelle, E. J., "Eigenvalue Similarity Rules for a Class of Rectangular Specially Orthotropic Laminated Plates," Paper presented at the Ninth U. S. National Congress of Applied Mechanics, June 1982.

[7]Brunelle, E. J., "The Fundamental Constants of Orthotropic Affine Plate/Slab Equations," *Proceedings of the 10th Annual Mini-Symposium on Aerospace Science and Technology*, March 1984; also *AIAA Journal*, Vol. 23, Dec. 1985, pp. 1957-1961.

[8]Brunelle, E. J., "Eigenvalue Similarity Rules for Symmetric Cross-Ply Laminated Plates," *Proceedings of 10th Annual Mini-Symposium on Aerospace Science and Technology*, March 1984; also *AIAA Journal*, Vol. 24, Jan. 1986, pp. 151-154.

[9]Brunelle, E. J. and Oyibo, G. A., "Generic Buckling Curves for Specially Orthotropic Rectangular Plates," *AIAA Journal*, Vol. 21, Aug. 1983, pp. 1150-1156.

[10]Oyibo, G. A. and Brunelle, E. J., "Vibrations of Circular Orthotropic Plates in Affine Space," *AIAA Journal*, Vol. 23, Feb. 1985, pp. 296-300.

[11]Donnell, L. H., *Beams, Plates, and Shells*, McGraw-Hill Book Co., New York, 1976, pp. 228-234.

[12]Tsai, S. W., "Structural Behavior of Composite Materials," NASA CR-71, July 1964.